Conceptual Questions for Review

Chapter 1

1.1 Which vectors are linear combinations of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$?

1.2 Compare the dot product of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in Question 1.2? What is the cosine of the angle between the x-axis and $\boldsymbol{v}$?

Chapter 2

2.1 Multiplying a matrix A times the column vector $\boldsymbol{x} = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $A\boldsymbol{x} = \boldsymbol{b}$ then the vector $\boldsymbol{b}$ is a linear combination of what vectors from the matrix A? In vector space language, $\boldsymbol{b}$ lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 6 \end{smallmatrix}\right]$ what are its pivots?

2.4 If A is the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 3 \end{smallmatrix}\right]$ find $\boldsymbol{b}$ and $\boldsymbol{c}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution and $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \begin{bmatrix} I & U \\ 0 & I \end{bmatrix}$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[A \quad I]$? If you solve the n equations Ax = columns of I then the solutions x are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

Chapter 3

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $Ax = b$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $Ax = b$.

3.9 If $Ax = b$ has exactly one solution for every b, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

3.13 Why is every row of A perpendicular to every vector in the nullspace?

3.14 How do you know that a column $\boldsymbol{u}$ times a row $\boldsymbol{v}^{\mathrm{T}}$ (both nonzero) has rank 1?

3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?

3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?

3.17 Describe a *pivot column* of A.

3.18 True? The vectors in the left nullspace of A have the form $A^{\mathrm{T}}\boldsymbol{y}$.

3.19 Why do the columns of every invertible matrix yield a basis?

Chapter 4

4.1 What does the word *complement* mean about orthogonal subspaces?

4.2 If $\boldsymbol{V}$ is a subspace of the 7-dimensional space $\mathbf{R}^7$, the dimensions of $\boldsymbol{V}$ and its orthogonal complement add to _____.

4.3 The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the vector _____.

4.4 The projection matrix onto the line through $\boldsymbol{a}$ is $P =$ _____.

4.5 The key equation to project $\boldsymbol{b}$ onto the column space of A is the *normal equation* _____.

4.6 The matrix $A^{\mathrm{T}}A$ is invertible when the columns of A are _____.

4.7 The least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ minimizes what error function?

4.8 What is the connection between the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ and the idea of projection onto the column space?

4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection $\boldsymbol{p}$ appear in the graph?

4.10 If the columns of Q are orthonormal, why is $Q^{\mathrm{T}}Q = I$?

4.11 What is the projection matrix P onto the columns of Q?

4.12 If Gram-Schmidt starts with the vectors $\boldsymbol{a} = (2,0)$ and $\boldsymbol{b} = (1,1)$, which two orthonormal vectors does it produce? If we keep $\boldsymbol{a} = (2,0)$ does Gram-Schmidt always produce the same two orthonormal vectors?

4.13 True? Every permutation matrix is an orthogonal matrix.

4.14 The inverse of the orthogonal matrix Q is _____.

Chapter 5

5.1 What is the determinant of the matrix $-I$?

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is $0, 0, 0, 3$. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering $(2, 5, 3, 4, 1)$ even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $I\boldsymbol{x} = \boldsymbol{b}$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $\boldsymbol{u} = (0, 0, 1)$ and $\boldsymbol{v} = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

Chapter 6

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[1\ \ 2\ \ 1]^{\mathrm{T}}[1\ \ 1\ \ 1]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

6.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A? Which might be larger?

6.9 Explain why the trace of AB equals the trace of BA.

6.10 How do the eigenvectors of A help to solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$?

6.11 How do the eigenvectors of A help to solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$?

6.12 Define the matrix exponential e^A and its inverse and its square.

6.13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?

6.14 What is the diagonalization formula when A is symmetric?

6.15 What does it mean to say that A is *positive definite*?

6.16 When is $B = A^{\mathrm{T}}A$ a positive definite matrix (A is real)?

6.17 If A is positive definite describe the surface $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = 1$ in $\mathbf{R}^n$.

6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B?

6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?

6.20 The SVD expresses A as a product of what three types of matrices?

6.21 How is the SVD for A linked to $A^{\mathrm{T}}A$?

Chapter 7

7.1 Define a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and give one example.

7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.

7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?

7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M?

7.5 Suppose our new basis is formed from the eigenvectors of a matrix A. What matrix represents A in this new basis?

7.6 If A and B are the matrices representing linear transformations S and T on $\mathbf{R}^n$, what matrix represents the transformation from $\boldsymbol{v}$ to $S(T(\boldsymbol{v}))$?

7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A?).

GLOSSARY: A DICTIONARY FOR LINEAR ALGEBRA

Adjacency matrix of a graph. Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ when edges go both ways (undirected).

Affine transformation $Tv = Av + v_0$ = linear transformation plus shift.

Associative Law $(AB)C = A(BC)$. Parentheses can be removed to leave ABC.

Augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the column space of A; then $[\,A \;\; \boldsymbol{b}\,]$ has the same rank as A. Elimination on $[\,A \;\; \boldsymbol{b}\,]$ keeps equations correct.

Back substitution. Upper triangular systems are solved in reverse order x_n to x_1.

Basis for V. Independent vectors $v_1, \ldots, v_d$ whose linear combinations give each vector in V as $v = c_1 v_1 + \ldots + c_d v_d$. V has many bases, each basis gives unique c's. A vector space has many bases!

Big formula for n **by** n **determinants.** Det(A) is a sum of $n!$ terms. For each term: Multiply one entry from each row and column of A: rows in order $1, \ldots, n$ and column order given by a permutation P. Each of the $n!$ P's has a + or − sign.

Block matrix. A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns. **Block multiplication** of AB is allowed if the block shapes permit.

Cayley-Hamilton Theorem. $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix.*

Change of basis matrix M. The old basis vectors v_j are combinations $\sum m_{ij} w_i$ of the new basis vectors. The coordinates of $c_1 v_1 + \cdots + c_n v_n = d_1 w_1 + \cdots + d_n w_n$ are related by $\boldsymbol{d} = M\boldsymbol{c}$. (For $n = 2$ set $v_1 = m_{11} w_1 + m_{21} w_2$, $v_2 = m_{12} w_1 + m_{22} w_2$.)

Characteristic equation $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A.

Cholesky factorization $A = C^{\mathrm{T}} C = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C. Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0 I + c_1 S + \cdots + c_{n-1} S^{n-1}$. $C\boldsymbol{x} =$ **convolution** $\boldsymbol{c} * \boldsymbol{x}$. Eigenvectors in F.

Cofactor C_{ij}. Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $A\boldsymbol{x} = \boldsymbol{b}$. The vector $\boldsymbol{b}$ becomes a combination of the columns of A. The system is solvable only when $\boldsymbol{b}$ is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ = space of all combinations of the columns of A.

Commuting matrices $AB = BA$. If diagonalizable, they share n eigenvectors.

Companion matrix. Put $c_1, \ldots, c_n$ in row n and put $n - 1$ ones just above the main diagonal. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots + c_n\lambda^{n-1} - \lambda^n)$.

Complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ to $A\boldsymbol{x} = \boldsymbol{b}$. (Particular $\boldsymbol{x}_p$) + ($\boldsymbol{x}_n$ in nullspace).

Complex conjugate $\overline{z} = a - ib$ for any complex number $z = a + ib$. Then $z\overline{z} = |z|^2$.

Condition number $cond(A) = c(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$. In $Ax = b$, the relative change $\|\delta x\|/\|x\|$ is less than $cond(A)$ times the relative change $\|\delta b\|/\|b\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method. A sequence of steps (end of Chapter 9) to solve positive definite $Ax = b$ by minimizing $\frac{1}{2}x^{\mathrm{T}}Ax - x^{\mathrm{T}}b$ over growing Krylov subspaces.

Covariance matrix Σ. When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means $\overline{x}_i$, the matrix Σ = mean of $(x - \overline{x})(x - \overline{x})^{\mathrm{T}}$ is positive (semi)definite; Σ is diagonal if the x_i are independent.

Cramer's Rule for $Ax = b$. B_j has b replacing column j of A; $x_j = \det B_j / \det A$

Cross product $u \times v$ **in** $\mathbf{R}^3$. Vector perpendicular to u and v, length $\|u\|\|v\||\sin\theta|$ = area of parallelogram, $u \times v$ = "determinant" of $[\,i \;\; j \;\; k\,;\; u_1 \;\; u_2 \;\; u_3\,;\; v_1 \;\; v_2 \;\; v_3\,]$.

Cyclic shift S. Permutation with $s_{21} = 1, s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are the nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$. Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$ and $|A^{-1}| = 1/|A|$ and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n - 1$, volume of box = $|\det(A)|$.

Diagonal matrix D. $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A. Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda$ = eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$. Λ = eigenvalue matrix and S = eigenvector matrix of A. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(V)$ = number of vectors in any basis for V.

Distributive Law $A(B + C) = AB + AC$. Add then multiply, or multiply then add.

Dot product = Inner product $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$. Complex dot product is $\overline{x}^{\mathrm{T}}y$. Perpendicular vectors have $\overline{x}^{\mathrm{T}}y = 0$. $(AB)_{ij}$ = (row i of A)$^{\mathrm{T}}$(column j of B).

Echelon matrix U. The first nonzero entry (the pivot) in each row comes in a later column than the pivot in the previous row. All zero rows come last.

Eigenvalue λ **and eigenvector** x. $Ax = \lambda x$ with $x \neq 0$ so $\det(A - \lambda I) = 0$.

Elimination. A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathrm{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix = Elementary matrix E_{ij}. The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $x^{\mathrm{T}}Ax = 1$. A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|x\| = 1$ the vectors $y = Ax$ lie on the ellipse $\|A^{-1}y\|^2 = y^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}u(0)$ solves $u' = Au$.

Factorization $A = LU$. If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT). A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n\boldsymbol{x}$ and $F_n^{-1}\boldsymbol{c}$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers $0, 1, 1, 2, 3, 5, \ldots$ satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1 + \sqrt{5})/2$ is the largest eigenvalue of the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$.

Four Fundamental Subspaces $\boldsymbol{C}(A), \boldsymbol{N}(A), \boldsymbol{C}(A^{\mathrm{T}}), \boldsymbol{N}(A^{\mathrm{T}})$. Use $\overline{A}^{\mathrm{T}}$ for complex A.

Fourier matrix F. Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}}F = nI$. Then $y = Fc$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A. Columns without pivots; these are combinations of earlier columns.

Free variable x_i. Column i has no pivot in elimination. We can give the $n - r$ free variables any values, then $A\boldsymbol{x} = \boldsymbol{b}$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $A\boldsymbol{x} = \boldsymbol{b}$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem. The nullspace $\boldsymbol{N}(A)$ and row space $\boldsymbol{C}(A^{\mathrm{T}})$ are orthogonal complements in $\mathbf{R}^n$(perpendicular from $A\boldsymbol{x} = \boldsymbol{0}$ with dimensions r and $n - r$). Applied to A^{T}, the column space $\boldsymbol{C}(A)$ is the orthogonal complement of $\boldsymbol{N}(A^{\mathrm{T}})$ in $\mathbf{R}^m$.

Gauss-Jordan method. Invert A by row operations on $[\,A \quad I\,]$ to reach $[\,I \quad A^{-1}\,]$.

Gram-Schmidt orthogonalization $A = QR$. Independent columns in A, orthonormal columns in Q. Each column $\boldsymbol{q}_j$ of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\mathrm{diag}(R) > \boldsymbol{0}$.

Graph G. Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n - 1$ edges and no closed loops.

Hankel matrix H. Constant along each antidiagonal; h_{ij} depends on $i + j$.

Hermitian matrix $A^{\mathrm{H}} = \overline{A}^{\mathrm{T}} = A$. Complex analog $\overline{a_{ji}} = a_{ij}$ of a symmetric matrix.

Hessenberg matrix H. Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix hilb(n). Entries $H_{ij} = 1/(i + j - 1) = \int_0^1 x^{i-1}x^{j-1}\,dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number: H is *ill-conditioned.*

Hypercube matrix P_L^2. Row $n + 1$ counts corners, edges, faces, ... of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**).** Diagonal entries $= 1$, off-diagonal entries $= 0$.

Incidence matrix of a directed graph. The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix. A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $v_1, \ldots, v_k$. No combination $c_1v_1 + \cdots + c_kv_k$ = zero vector unless all $c_i = 0$. If the v's are the columns of A, the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$.

Inverse matrix A^{-1}. Square matrix with $A^{-1}A = I$ and $AA^{-1} = I$. No inverse if $\det A = 0$ and $\operatorname{rank}(A) < n$ and $Ax = \mathbf{0}$ for a nonzero vector x. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$. Cofactor formula $(A^{-1})_{ij} = C_{ji}/\det A$.

Iterative method. A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$. If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \operatorname{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1's on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector.

Kirchhoff's Laws. *Current Law*: net current (in minus out) is zero at each node. *Voltage Law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$. Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, b)$. The subspace spanned by $b, Ab, \ldots, A^{j-1}b$. Numerical methods approximate $A^{-1}b$ by x_j with residual $b - Ax_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least squares solution $\widehat{x}$. The vector $\widehat{x}$ that minimizes the error $\|e\|^2$ solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$. Then $e = b - A\widehat{x}$ is orthogonal to all columns of A.

Left inverse A^+. If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $N(A^{\mathrm{T}})$. Nullspace of A^{T} = "left nullspace" of A because $y^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$.

Length $\|x\|$. Square root of $x^{\mathrm{T}}x$ (Pythagoras in n dimensions).

Linear combination $cv + dw$ or $\sum c_jv_j$. Vector addition and scalar multiplication.

Linear transformation T. Each vector v in the input space transforms to $T(v)$ in the output space, and linearity requires $T(cv + dw) = c\,T(v) + d\,T(w)$. Examples: Matrix multiplication Av, differentiation and integration in function space.

Linearly dependent $v_1, \ldots, v_n$. A combination other than all $c_i = 0$ gives $\sum c_iv_i = \mathbf{0}$.

Lucas numbers $L_n = 2, 1, 3, 4, \ldots$ satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_2^n$, with $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ from the Fibonacci matrix $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]$. Compare $L_0 = 2$ with $F_0 = 0$.

Markov matrix M. All $m_{ij} \geq 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady state eigenvector $Ms = s > \mathbf{0}$.

Matrix multiplication AB. The i, j entry of AB is (row i of A)·(column j of B) = $\sum a_{ik}b_{kj}$. By columns: Column j of AB = A times column j of B. By rows: row i of A multiplies B. Columns times rows: AB = sum of (column k)(row k). All these equivalent definitions come from the rule that AB times x equals A times Bx.

Minimal polynomial of A. The lowest degree polynomial with $m(A)$ = zero matrix. This is $p(\lambda) = \det(A - \lambda I)$ if no eigenvalues are repeated; always $m(\lambda)$ divides $p(\lambda)$.

Multiplication $Ax = x_1$(column 1) $+ \cdots + x_n$(column n) = combination of columns.

Multiplicities AM and GM. The algebraic multiplicity AM of λ is the number of times λ appears as a root of $\det(A - \lambda I) = 0$. The geometric multiplicity GM is the number of independent eigenvectors for λ (= dimension of the eigenspace).

Multiplier ℓ_{ij}. The pivot row j is multiplied by ℓ_{ij} and subtracted from row i to eliminate the i, j entry: ℓ_{ij} = (entry to eliminate) / (jth pivot).

Network. A directed graph that has constants $c_1, \ldots, c_m$ associated with the edges.

Nilpotent matrix N. Some power of N is the zero matrix, $N^k = 0$. The only eigenvalue is $\lambda = 0$ (repeated n times). Examples: triangular matrices with zero diagonal.

Norm $\|A\|$. The "$\boldsymbol{\ell}^2$ norm" of A is the maximum ratio $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\| = \sigma_{\max}$. Then $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$ and $\|AB\| \le \|A\|\|B\|$ and $\|A + B\| \le \|A\| + \|B\|$. **Frobenius norm** $\|A\|_F^2 = \sum\sum a_{ij}^2$. The $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^\infty$ norms are largest column and row sums of $|a_{ij}|$.

Normal equation $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Gives the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ if A has full rank n (independent columns). The equation says that (columns of A)$\cdot(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = 0$.

Normal matrix. If $NN^{\mathrm{T}} = N^{\mathrm{T}}N$, then N has orthonormal (complex) eigenvectors.

Nullspace $N(A)$ = All solutions to $A\boldsymbol{x} = \boldsymbol{0}$. Dimension $n - r$ = (# columns) − rank.

Nullspace matrix N. The columns of N are the $n - r$ special solutions to $A\boldsymbol{s} = \boldsymbol{0}$.

Orthogonal matrix Q. Square matrix with orthonormal columns, so $Q^{\mathrm{T}} = Q^{-1}$. Preserves length and angles, $\|Q\boldsymbol{x}\| = \|\boldsymbol{x}\|$ and $(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$. All $|\lambda| = 1$, with orthogonal eigenvectors. Examples: Rotation, reflection, permutation.

Orthogonal subspaces. Every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$.

Orthonormal vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$. Dot products are $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$ if $i \ne j$ and $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_i = 1$. The matrix Q with these orthonormal columns has $Q^{\mathrm{T}}Q = I$. If $m = n$ then $Q^{\mathrm{T}} = Q^{-1}$ and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ is an **orthonormal basis** for $\mathbf{R}^n$: every $\boldsymbol{v} = \sum(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{q}_j)\boldsymbol{q}_j$.

Outer product $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row = rank one matrix.

Partial pivoting. In each column, choose the largest available pivot to control roundoff; all multipliers have $|\ell_{ij}| \le 1$. See *condition number*.

Particular solution $\boldsymbol{x}_p$. Any solution to $A\boldsymbol{x} = \boldsymbol{b}$; often $\boldsymbol{x}_p$ has free variables = 0.

Pascal matrix P_S = pascal(n) = the symmetric matrix with binomial entries $\binom{i+j-2}{i-1}$. $P_S = P_L P_U$ all contain Pascal's triangle with det = 1 (see Pascal in the index).

Permutation matrix P. There are $n!$ orders of $1, \ldots, n$. The $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is *even* or *odd* ($\det P = 1$ or -1) based on the number of row exchanges to reach I.

Pivot columns of A. Columns that contain pivots after row reduction. These are *not* combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot. The diagonal entry (*first nonzero*) at the time when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$. Vectors $\boldsymbol{x}$ with $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = 0$. Plane is perpendicular to $\boldsymbol{a} \ne \boldsymbol{0}$.

Polar decomposition $A = QH$. Orthogonal Q times positive (semi)definite H.

Positive definite matrix A. Symmetric matrix with positive eigenvalues and positive pivots. *Definition:* $x^TAx > 0$ unless $x = 0$. Then $A = LDL^T$ with diag$(D) > 0$.

Projection $p = a(a^Tb/a^Ta)$ onto the line through a. $P = aa^T/a^Ta$ has rank 1.

Projection matrix P onto subspace S. Projection $p = Pb$ is the closest point to b in S, error $e = b - Pb$ is perpendicular to S. $P^2 = P = P^T$, eigenvalues are 1 or 0, eigenvectors are in S or $S^\perp$. If columns of A = basis for S then $P = A(A^TA)^{-1}A^T$.

Pseudoinverse A^+ (Moore-Penrose inverse). The n by m matrix that "inverts" A from column space back to row space, with $N(A^+) = N(A^T)$. A^+A and AA^+ are the projection matrices onto the row space and column space. Rank(A^+) = rank(A).

Random matrix rand(n) or randn(n). MATLAB creates a matrix with random entries, uniformly distributed on $[\,0\ \ 1\,]$ for rand and standard normal distribution for randn.

Rank one matrix $A = uv^T \neq 0$. Column and row spaces = lines cu and cv.

Rank $r(A)$ = number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(x) = x^TAx/x^Tx$ for symmetric A: $\lambda_{\min} \le q(x) \le \lambda_{\max}$. Those extremes are reached at the eigenvectors x for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form R = rref(A). Pivots = 1; zeros above and below pivots; the r nonzero rows of R give a basis for the row space of A.

Reflection matrix (Householder) $Q = I - 2uu^T$. Unit vector u is reflected to $Qu = -u$. All x in the plane mirror $u^Tx = 0$ have $Qx = x$. Notice $Q^T = Q^{-1} = Q$.

Right inverse A^+. If A has full row rank m, then $A^+ = A^T(AA^T)^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ rotates the plane by θ and $R^{-1} = R^T$ rotates back by $-\theta$. Eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors are $(1, \pm i)$. $c, s = \cos\theta, \sin\theta$.

Row picture of $Ax = b$. Each equation gives a plane in $\mathbf{R}^n$; the planes intersect at x.

Row space $C(A^T)$ = all combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$. A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f/\partial x_i \partial x_j$ = **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$. Appears in block elimination on $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Schwarz inequality $|v \cdot w| \le \|v\|\,\|w\|$.Then $|v^TAw|^2 \le (v^TAv)(w^TAw)$ for pos def A.

Semidefinite matrix A. (Positive) semidefinite: all $x^TAx \ge 0$, all $\lambda \ge 0$; A = any R^TR.

Similar matrices A **and** B. Every $B = M^{-1}AM$ has the same eigenvalues as A.

Simplex method for linear programming. The minimum cost vector x^* is found by moving from corner to lower cost corner along the edges of the feasible set (where the constraints $Ax = b$ and $x \ge 0$ are satisfied). Minimum cost at a corner!

Singular matrix A. A square matrix that has no inverse: det$(A) = 0$.

Singular Value Decomposition (SVD) $A = U\Sigma V^T$ = **(orthogonal)(diag)(orthogonal)** First r columns of U and V are orthonormal bases of $C(A)$ and $C(A^T)$, $Av_i = \sigma_i u_i$ with singular value $\sigma_i > 0$. Last columns are orthonormal bases of nullspaces.

Skew-symmetric matrix K. The transpose is $-K$, since $K_{ij} = -K_{ji}$. Eigenvalues are pure imaginary, eigenvectors are orthogonal, e^{Kt} is an orthogonal matrix.

Solvable system $Ax = b$. The right side b is in the column space of A.

Spanning set. Combinations of $v_1, \ldots, v_m$ fill the space. The columns of A span $C(A)$!

Special solutions to $As = 0$. One free variable is $s_i = 1$, other free variables $= 0$.

Spectral Theorem $A = Q\Lambda Q^T$. Real symmetric A has real λ's and orthonormal q's.

Spectrum of A = the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. **Spectral radius** = max of $|\lambda_i|$.

Standard basis for $\mathbf{R}^n$. Columns of n by n identity matrix (written i, j, k in $\mathbf{R}^3$).

Stiffness matrix If x gives the movements of the nodes, Kx gives the internal forces. $K = A^TCA$ where C has spring constants from Hooke's Law and Ax = stretching.

Subspace S **of** V. Any vector space inside V, including V and Z = {zero vector only}.

Sum $V + W$ **of subspaces.** Space of all (v in V) + (w in W). **Direct sum**: $V \cap W = \{0\}$.

Symmetric factorizations $A = LDL^T$ **and** $A = Q\Lambda Q^T$. Signs in Λ = signs in D.

Symmetric matrix A. The transpose is $A^T = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric.

Toeplitz matrix. Constant down each diagonal = time-invariant (shift-invariant) filter.

Trace of A = sum of diagonal entries = sum of eigenvalues of A. Tr AB = Tr BA.

Transpose matrix A^T. Entries $A^T_{ij} = A_{ji}$. A^T is n by m, A^TA is square, symmetric, positive semidefinite. The transposes of AB and A^{-1} are B^TA^T and $(A^T)^{-1}$.

Triangle inequality $\|u + v\| \le \|u\| + \|v\|$. For matrix norms $\|A + B\| \le \|A\| + \|B\|$.

Tridiagonal matrix T: $t_{ij} = 0$ if $|i - j| > 1$. T^{-1} has rank 1 above and below diagonal.

Unitary matrix $U^H = \overline{U}^T = U^{-1}$. Orthonormal columns (complex analog of Q).

Vandermonde matrix V. $Vc = b$ gives coefficients of $p(x) = c_0 + \cdots + c_{n-1}x^{n-1}$ with $p(x_i) = b_i$. $V_{ij} = (x_i)^{j-1}$ and det V = product of $(x_k - x_i)$ for $k > i$.

Vector v **in** $\mathbf{R}^n$. Sequence of n real numbers $v = (v_1, \ldots, v_n)$ = point in $\mathbf{R}^n$.

Vector addition. $v + w = (v_1 + w_1, \ldots, v_n + w_n)$ = diagonal of parallelogram.

Vector space V. Set of vectors such that all combinations $cv + dw$ remain within V. Eight required rules are given in Section 3.1 for scalars c, d and vectors v, w.

Volume of box. The rows (or the columns) of A generate a box with volume $|\det(A)|$.

Wavelets $w_{jk}(t)$. Stretch and shift the time axis to create $w_{jk}(t) = w_{00}(2^j t - k)$.

MATRIX FACTORIZATIONS

1. $A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1's on the diagonal} \end{pmatrix}$

   Requirements: No row exchanges. The pivots in D are divided out to leave 1's on the diagonal of U. If A is symmetric then U is L^{T} and $A = LDL^{\mathrm{T}}$.

3. $PA = LU$ (permutation matrix P to avoid zeros in the pivot positions).

   Requirements: A is invertible. Then P, L, U are invertible. P does all of the row exchanges in advance, to allow normal LU. Alternative: $A = L_1 P_1 U_1$.

4. $EA = R$ (m by m invertible E) (any matrix A) = rref(A).

   Requirements: None! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The last $m - r$ rows of E are a basis for the left nullspace of A; they multiply A to give zero rows in R. The first r columns of E^{-1} are a basis for the column space of A.

5. $A = C^{\mathrm{T}}C$ = (lower triangular) (upper triangular) with $\sqrt{D}$ on both diagonals

   Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* C = chol(A) has $C^{\mathrm{T}} = L\sqrt{D}$, so $C^{\mathrm{T}}C = LDL^{\mathrm{T}}$.

6. $A = QR$ = (orthonormal columns in Q) (upper triangular R).

   Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt or Householder process. If A is square then $Q^{-1} = Q^{\mathrm{T}}$.

7. $A = S\Lambda S^{-1}$ = (eigenvectors in S) (eigenvalues in Λ) (left eigenvectors in S^{-1}).

   Requirements: A must have n linearly independent eigenvectors.

8. $A = Q\Lambda Q^{\mathrm{T}}$ = (orthogonal matrix Q) (real eigenvalue matrix Λ) (Q^{T} is Q^{-1}).

   Requirements: A is *real and symmetric*. This is the Spectral Theorem.

9. $A = MJM^{-1}$ = (generalized eigenvectors in M) (Jordan blocks in J) (M^{-1}).

Requirements: A is any square matrix. This *Jordan form* J has a block for each independent eigenvector of A. Every block has only one eigenvalue.

10. $$A = U\Sigma V^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ U \text{ is } m \times n \end{pmatrix} \begin{pmatrix} m \times n \text{ singular value matrix} \\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ V \text{ is } n \times n \end{pmatrix}.$$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of AA^{T} in U and eigenvectors of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$.

11. $$A^+ = V\Sigma^+U^{\mathrm{T}} = \begin{pmatrix} \text{orthogonal} \\ n \times n \end{pmatrix} \begin{pmatrix} n \times m \text{ pseudoinverse of } \Sigma \\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal} \end{pmatrix} \begin{pmatrix} \text{orthogonal} \\ m \times m \end{pmatrix}.$$

Requirements: None. The *pseudoinverse* A^+ has A^+A = projection onto row space of A and AA^+ = projection onto column space. The shortest least-squares solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = A^+\boldsymbol{b}$. This solves $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

12. $A = QH$ = (orthogonal matrix Q) (symmetric positive definite matrix H).

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD.

13. $A = U\Lambda U^{-1}$ = (unitary U) (eigenvalue matrix Λ) (U^{-1} which is $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}}$).

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$: Hermitian case.

14. $A = UTU^{-1}$ = (unitary U) (triangular T with λ's on diagonal) ($U^{-1} = U^{\mathrm{H}}$).

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular: Section 6.4.

15. $$F_n = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} & \\ & F_{n/2} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix} = \text{one step of the (recursive) } \textbf{FFT}.$$

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$: $F_n\overline{F}_n = nI$. D has $1, w, \ldots, w^{n/2-1}$ on its diagonal. For $n = 2^\ell$ the *Fast Fourier Transform* will compute $F_n x$ with only $\frac{1}{2}n\ell = \frac{1}{2}n \log_2 n$ multiplications from ℓ stages of D's.

MATLAB TEACHING CODES

These Teaching Codes are directly available from **web.mit.edu/ 18.06**

cofactor	Compute the n by n matrix of cofactors.
cramer	Solve the system $Ax = b$ by Cramer's Rule.
deter	Matrix determinant computed from the pivots in $PA = LU$.
eigen2	Eigenvalues, eigenvectors, and $\det(A - \lambda I)$ for 2 by 2 matrices.
eigshow	Graphical demonstration of eigenvalues and singular values.
eigval	Eigenvalues and their multiplicity as roots of $\det(A - \lambda I) = 0$.
eigvec	Compute as many linearly independent eigenvectors as possible.
elim	Reduction of A to row echelon form R by an invertible E.
findpiv	Find a pivot for Gaussian elimination (used by **plu**).
fourbase	Construct bases for all four fundamental subspaces.
grams	Gram-Schmidt orthogonalization of the columns of A.
house	2 by 12 matrix giving corner coordinates of a house.
inverse	Matrix inverse (if it exists) by Gauss-Jordan elimination.
leftnull	Compute a basis for the left nullspace.
linefit	Plot the least squares fit to m given points by a line.
lsq	Least squares solution to $Ax = b$ from $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.
normal	Eigenvalues and orthonormal eigenvectors when $A^{\mathrm{T}}A = AA^{\mathrm{T}}$.
nulbasis	Matrix of special solutions to $Ax = 0$ (basis for nullspace).
orthcomp	Find a basis for the orthogonal complement of a subspace.
partic	Particular solution of $Ax = b$, with all free variables zero.
plot2d	Two-dimensional plot for the house figures.
plu	Rectangular $PA = LU$ factorization with row exchanges.
poly2str	Express a polynomial as a string.
project	Project a vector b onto the column space of A.
projmat	Construct the projection matrix onto the column space of A.
randperm	Construct a random permutation.
rowbasis	Compute a basis for the row space from the pivot rows of R.
samespan	Test whether two matrices have the same column space.
signperm	Determinant of the permutation matrix with rows ordered by p.
slu	LU factorization of a square matrix using *no row exchanges*.
slv	Apply **slu** to solve the system $Ax = b$ allowing no row exchanges.
splu	Square $PA = LU$ factorization *with row exchanges*.
splv	The solution to a square, invertible system $Ax = b$.
symmeig	Compute the eigenvalues and eigenvectors of a symmetric matrix.
tridiag	Construct a tridiagonal matrix with constant diagonals a, b, c.

Index

With the single heading* "Matrix" *this page indexes the active life of linear algebra.

Index of Symbols

Linear Algebra Websites

math.mit.edu/linearalgebra Dedicated to help readers and teachers working with this book

ocw.mit.edu MIT's OpenCourseWare site including video lectures in 18.06 and 18.085-6

web.mit.edu/18.06 Current and past exams and homeworks with extra materials

wellesleycambridge.com Ordering information for books by Gilbert Strang

LINEAR ALGEBRA IN A NUTSHELL

((The matrix A is n by n))

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$A\boldsymbol{x} = \mathbf{0}$ has one solution $\boldsymbol{x} = \mathbf{0}$	$A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions
$A\boldsymbol{x} = \boldsymbol{b}$ has one solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$	$A\boldsymbol{x} = \boldsymbol{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r = n$	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

This book is designed to help students understan
central problems of linear algebra:

$Ax = b$	n by n	Chapters 1-2	Linear systems
$Ax = b$	m by n	Chapters 3-4	Least squares
$Ax = \lambda x$	n by n	Chapters 5-6	Eigenvalues
$Av = \sigma u$	m by n	Chapters 6-7	Singular values

The diagram on the front cover shows the four fundamental subspaces for the matrix A. Those subspaces lead to the Fundamental Theorem of Linear Algebra:

1. The dimensions of the four subspaces
2. The orthogonality of the two pairs
3. The best bases for all four subspaces

This is the textbook that accompanies the author's video lectures and the review material on MIT's OpenCourseWare.

ocw.mit.edu* and *web.mit.edu/18.06

Many universities and colleges (and now high schools) use this textbook. Chapters 7-10 are for a second course on linear algebra.

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